Exercise 59

Determine whether $f'(0)$ exists.

$$f(x) = \begin{cases} x \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Solution

Recall that the derivative of f at $x = a$ is defined by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Plug in $a = 0$.

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{h \sin \frac{1}{h} - 0}{h}$$

$$= \lim_{h \to 0} \frac{h \sin \frac{1}{h}}{h}$$

$$= \lim_{h \to 0} \sin \frac{1}{h}$$

This limit does not exist, so $f'(0)$ does not exist.